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Multivariate fractionally integrated CARMA processes

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Abstract

A multivariate analogue of the fractionally integrated continuous time autoregressive moving average (FICARMA) process defined by Brockwell [Representations of continuous-time ARMA processes, J. Appl. Probab. 41 (A) (2004) 375–382] is introduced. We show that the multivariate FICARMA process has two kernel representations: as an integral over the fractionally integrated CARMA kernel with respect to a Lévy process and as an integral over the original (not fractionally integrated) CARMA kernel with respect to the corresponding fractional Lévy process (FLP). In order to obtain the latter representation we extend FLPs to the multivariate setting. In particular we give a spectral representation of FLPs and consequently, derive a spectral representation for FICARMA processes. Moreover, various probabilistic properties of the multivariate FICARMA process are discussed. As an example we consider multivariate fractionally integrated Ornstein–Uhlenbeck processes.

AMS 1991 subject classification: primary 60G12; secondary 60G20;60H05

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1. Introduction

Continuous time models for multivariate time series are of considerable interest, especially, when dealing with data observed at irregularly spaced time points or high-frequency data, as they appear in finance, economics or telecommunications.

Being the continuous time analogue of the well-known autoregressive moving average (ARMA) processes (see e.g. [8]), Lévy-driven continuous time ARMA (CARMA) processes, have been extensively studied over the last years (see e.g. [5,6,26] and references therein). Recently,

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multivariate CARMA (MCARMA) processes have been developed and studied by [18]. CARMA and thus MCARMA processes are short memory moving average processes and hence their autocorrelation functions show an exponential rate of decrease. However, often observed time series show long memory behavior in the sense that they seem to require models, whose autocorrelation functions follow a power law and where the decay is so slow that the autocorrelations are not summable.

Aiming at long memory models, using a fractional integration of the CARMA kernel, [7] defined Lévy-driven fractionally integrated CARMA (FICARMA) processes, where the autocorrelations are hyperbolically decaying. An alternative approach, which leads to the same class of long memory processes, was discussed in [17], where the so-called fractional Lévy processes (FLPs) were introduced and used to generate long memory. Processes generated by fractional integration are the most widely used long memory time series in economics and econometrics. A survey of applications of fractional integration and long memory in macroeconomics and finance is [14]. Furthermore, by considering Lévy-driven fractional processes, instead of Gaussian fractional processes, one obtains a much richer class of possibly heavy-tailed stationary processes with many potential applications in finance, where such heavy tails are frequently observed in practice.

So far only univariate Lévy-driven FICARMA processes have been defined and investigated. However, in order to model the joint behavior of several time series (e.g. prices of various stocks) multivariate models are required. Our aim in this paper is to define Lévy-driven multivariate FICARMA (MFICARMA) processes and study their probabilistic properties, where we follow two approaches. The first one is based on a fractional integration of the CARMA kernel, whereas the second approach substitutes the driving Lévy process by the corresponding FLP and leads to the same L^2-process. We thus obtain a model which is the continuous time analogue of the well-known multivariate fractionally integrated ARMA model (see e.g. [8] or [23] and the references therein) as well as the multivariate analogue of the univariate FICARMA processes studied by [9]. In particular, we obtain a spectral representation of FLPs which allows us to develop a spectral representation of MFICARMA processes. This is a new result which has not been given for (univariate) FICARMA processes, yet.

The paper is organized as follows. Section 2 contains the preliminaries. We review elementary properties of multivariate Lévy processes in Section 2.1 and the stochastic integration theory for deterministic functions with respect to them in Section 2.2. The extension of FLPs to the multivariate setting is given in Section 2.3. Since, depending on the driving Lévy process, FLPs are not always semimartingales, stochastic integration is not straightforward. We consider the integration theory with respect to multivariate FLPs in Section 2.4. A fundamental result is obtained in Section 2.5, namely a spectral representation for FLPs and a spectral representation for integrals with respect to FLPs. This result may have interest of its own and is later used to obtain a spectral representation for MFICARMA processes. We conclude Section 2 with a brief summary of Lévy-driven multivariate CARMA processes, recently introduced by [18]. Based on their results, in Section 3, a multivariate analogue of the FICARMA process [7] is developed. We show in Section 3.1 that the multivariate FICARMA process has two kernel representations: (I) as an integral over the fractionally integrated CARMA kernel with respect to a Lévy process and (II) as an integral over the original (not fractionally integrated) CARMA kernel with respect to the corresponding FLP. We would like to emphasize that both MFICARMA representations lead to the same L^2-process. However, the first representation is useful to derive distributional properties, whereas it is the second one, that enables us to obtain a spectral representation of FICARMA processes. Furthermore, we derive probabilistic properties of MFICARMA models. In particular, we characterize the characteristic triplet, the stationary limiting distribution, the covariance matrix function and the spectral density. Moreover, we investigate the sample path behavior and give conditions for the existence of a C_b^∞ density. As an example we consider in Section 4 the fractional Ornstein–Uhlenbeck process.

Throughout this paper we use the following notation. We call $M_m(\mathbb{R})$ the space of all real $m \times m$-matrices and let A^T and A^* denote the transposed and adjoint, respectively, of the matrix A. Furthermore, $I_m \in M_m(\mathbb{R})$ is the identity matrix and $\|A\|$ is the operator norm of $A \in M_m(\mathbb{R})$ corresponding to the norm $\|x\|$ for $x \in \mathbb{R}^m$. $I_B(\cdot)$ is the indicator function of the set B and we write a.s. if something holds almost surely. Moreover, we set $\mathbb{R}_0^m := \mathbb{R}^m \setminus \{0\}$ and throughout assume as given an underlying complete, filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geqslant 0}, P)$ with right continuous filtration $(\mathscr{F}_t)_{t\geqslant 0}$. Finally, we define for $p > 0$,

$$L^p(M_m(\mathbb{R})) := \left\{ f : \mathbb{R} \times \mathbb{R} \to M_m(\mathbb{R}),\ \int_{\mathbb{R}} \|f(t,s)\|^p \, ds < \infty, \quad \text{for all } t \in \mathbb{R} \right\}.$$

Notice that the space $L^p(M_m(\mathbb{R}))$ is independent of the norm $\|\cdot\|$ on $M_m(\mathbb{R})$ used in the definition.

2. Preliminaries

2.1. Basic facts on multivariate Lévy processes

We state some elementary properties of multivariate Lévy processes that will be needed below. For a more general treatment and proofs we refer to [21] and [24].

We consider a Lévy process $\mathbf{L} = \{\mathbf{L}(t)\}_{t\geqslant 0}$ in $\mathbb{R}^m$ without Brownian component determined by its characteristic function in the Lévy–Khintchine form $E\left[e^{i\langle u, \mathbf{L}(t)\rangle}\right] = \exp\{t\psi(u)\}, \quad t \geqslant 0$, where

$$\psi(u) = i\langle \gamma, u\rangle + \int_{\mathbb{R}^m} (e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle I_{\{\|x\|\leqslant 1\}})\, \nu(dx), \quad u \in \mathbb{R}^m, \tag{2.1}$$

where $\gamma \in \mathbb{R}^m$ and ν satisfies $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^m} (\|x\|^2 \wedge 1)\, \nu(dx) < \infty$.

The measure ν is the Lévy measure of $\mathbf{L}$. We assume that ν satisfies additionally

$$\int_{\|x\|>1} \|x\|^2 \, \nu(dx) < \infty, \tag{2.2}$$

i.e. $\mathbf{L}$ has finite mean and covariance matrix function Σ_L given by

$$\Sigma_L = \int_{\mathbb{R}^m} xx^T \, \nu(dx). \tag{2.3}$$

We restrict ourselves to the case where $E[\mathbf{L}(1)] = 0$. From (2.2) and $E[\mathbf{L}(1)] = 0$ follows that $\gamma = -\int_{\|x\|>1} x \, \nu(dx)$ and (2.1) can be written in the form

$$\psi(u) = \int_{\mathbb{R}^m} (e^{i\langle u,x\rangle} - 1 - i\langle u, x\rangle)\, \nu(dx), \quad u \in \mathbb{R}^m. \tag{2.4}$$

It is a well-known fact that to every Lévy process $\mathbf{L}$ on $\mathbb{R}^m$ one can associate a random measure on $\mathbb{R}_0^m \times \mathbb{R}$ describing the jumps of $\mathbf{L}$. For any measurable set $B \subset \mathbb{R}_0^m \times \mathbb{R}$,

$$J(B) = \sharp\{s \in \mathbb{R} : \ (\mathbf{L}(s) - \mathbf{L}(s-), s) \in B\}. \tag{2.5}$$

The jump measure J is a Poisson random measure on $\mathbb{R}_0^m \times \mathbb{R}$ (see e.g. Definition 2.18 in [12]) with intensity measure $n(dx, ds) = \nu(dx)\, ds$. By the Lévy-Itô decomposition we can then rewrite $\mathbf{L}$ a.s. as

$$\mathbf{L}(t) = \int_{x\in\mathbb{R}_0^m,\, s\in[0,t]} x\,\tilde{J}(dx, ds), \quad t \geqslant 0. \tag{2.6}$$

Here $\tilde{J}(dx, ds) = J(dx, ds) - \nu(dx)\, ds$ is the compensated jump measure. Moreover, $\mathbf{L}$ is a martingale.

Throughout this paper we will work with a two-sided Lévy process $\mathbf{L} = \{\mathbf{L}(t)\}_{t\in\mathbb{R}}$ constructed by taking two independent copies $\{\mathbf{L}_1(t)\}_{t\geqslant 0}$, $\{\mathbf{L}_2(t)\}_{t\geqslant 0}$ of a one-sided Lévy process and setting

$$\mathbf{L}(t) = \begin{cases} \mathbf{L}_1(t) & \text{if } t \geqslant 0, \\ -\mathbf{L}_2(-t-) & \text{if } t < 0. \end{cases} \tag{2.7}$$

2.2. Stochastic integrals with respect to Lévy processes

In this section we consider the stochastic process $\mathbf{X} = \{\mathbf{X}(t)\}_{t\in\mathbb{R}}$ in $\mathbb{R}^m$ given by

$$\mathbf{X}(t) = \int_{\mathbb{R}} f(t, s)\, \mathbf{L}(ds), \quad t \in \mathbb{R}, \tag{2.8}$$

where $f : \mathbb{R} \times \mathbb{R} \to M_m(\mathbb{R})$ is a measurable function and $\mathbf{L} = \{\mathbf{L}(t)\}_{t\in\mathbb{R}}$ is an m-dimensional Lévy process without Brownian component. Again, we would like to stress that throughout this paper we assume a Lévy process $\mathbf{L}$ which satisfies $E[\mathbf{L}(1)] = 0$ and $E[\mathbf{L}(1)\mathbf{L}(1)^T] < \infty$, i.e. $\mathbf{L}$ can be represented as in (2.6) together with (2.7).

In this case it is a well-known fact, that the process $\mathbf{X}$ in (2.8) can be represented by

$$\mathbf{X}(t) = \int_{\mathbb{R}_0^m\times\mathbb{R}} f(t, s)x\, \tilde{J}(dx, ds), \quad t \in \mathbb{R}, \tag{2.9}$$

where $\tilde{J}(dx, ds) = J(dx, ds) - \nu(dx)\, ds$ is the compensated jump measure of $\mathbf{L}$. If $f(t, \cdot) \in L^2(M_m(\mathbb{R}))$, the stochastic integral (2.9) exists in $L^2(\Omega, P)$. Then

$$E\left[\mathbf{X}(t)\mathbf{X}(t)^T\right] = \int_{\mathbb{R}} f(t, s)\Sigma_L f^T(t, s)\, ds, \quad t \in \mathbb{R}, \tag{2.10}$$

and the law of $\mathbf{X}(t)$ is for all $t \in \mathbb{R}$ infinitely divisible with characteristic function

$$E\left[\exp\left\{i\langle u, \mathbf{X}(t)\rangle\right\}\right] = \exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}^m} \left(e^{i\langle u, f(t,s)x\rangle} - 1 - i\langle u, f(t, s)x\rangle\right) \nu(dx)\, ds\right\} \tag{2.11}$$

(see e.g. [22,16,25]).

2.3. Multivariate FLPs

FLPs were introduced in [17] by replacing the Brownian motion in the moving average representation of fractional Brownian motion by a Lévy processes without Gaussian part, having zero mean and finite second moments. Here we extend the definition of a univariate FLP to the multivariate setting. For details on univariate FLPs we refer to [17].

Definition 2.1 (*MFLP*). For fractional integration parameter $d = (d_1, \ldots, d_m)^T$ such that $0 < d_j < 0.5$ for all $j = 1, \ldots, m$, we define the kernel $f_t : \mathbb{R} \to M_m(\mathbb{R})$ by

$$f_t(s) := \begin{pmatrix} \frac{1}{\Gamma(d_1+1)}[(t-s)_+^{d_1} - (-s)_+^{d_1}] & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\Gamma(d_m+1)}[(t-s)_+^{d_m} - (-s)_+^{d_m}] \end{pmatrix},$$
$$s, t \in \mathbb{R}. \tag{2.12}$$

Then we define a multivariate fractional Lévy process (MFLP) by

$$\mathbf{M}_d(t) = (M_{d_1}(t), \ldots, M_{d_m}(t))^T = \int_{\mathbb{R}} f_t(s)\,\mathbf{L}(ds), \quad t \in \mathbb{R}, \tag{2.13}$$

where $\mathbf{L}(t) = (L^1(t), \ldots, L^m(t))^T$ and $L^j = \{L^j(t)\}_{t\in\mathbb{R}}$, $j = 1, \ldots, m$ are Lévy processes without Gaussian component on $\mathbb{R}$ satisfying $E[L^j(1)] = 0$ and $E[L^j(1)^2] < \infty$, $j = 1, \ldots m$.

Note that $f_t \in L^2(M_m(\mathbb{R}))$ and therefore the following proposition is an obvious consequence of (2.11).

Proposition 2.2. *The process* $\{\mathbf{M}_d(t)\}_{t\in\mathbb{R}}$ *given in* (2.13) *is well-defined in* $L^2(\Omega, P)$. *The distribution of* $\mathbf{M}_d(t)$ *is infinitely divisible with characteristic triplet* $(\gamma_M^t, 0, \nu_M^t)$, *where*

$$\gamma_M^t = -\int_{\mathbb{R}}\int_{\mathbb{R}^m} f_t(s)x I_{\{\|f_t(s)x\|>1\}}\,\nu(dx)\,ds \quad \text{and} \tag{2.14}$$

$$\nu_M^t(B) = \int_{\mathbb{R}}\int_{\mathbb{R}^m} I_B(f_t(s)x)\,\nu(dx)\,ds. \tag{2.15}$$

Furthermore, for $t \in \mathbb{R}$ *and* $z \in \mathbb{R}^m$,

$$E[\exp i\langle z, \mathbf{M}_d(t)\rangle] = \exp\left\{\int_{\mathbb{R}}\int_{\mathbb{R}^m}\left(e^{i\langle z, f_t(s)x\rangle} - 1 - i\langle z, f_t(s)x\rangle\right)\nu(dx)\,ds\right\}. \tag{2.16}$$

Remark 2.3. The process $\mathbf{M}_d$ is a.s. equal to an improper Riemann integral as for its *j*th component we have

$$M_{d_j}(t) = \frac{1}{\Gamma(d_j)}\int_{\mathbb{R}}[(t-s)_+^{d_j-1} - (-s)_+^{d_j-1}]\,L^j(s)\,ds, \quad t \in \mathbb{R}. \tag{2.17}$$

Moreover, (2.17) is continuous in *t* (see [17]).

Using (2.10), we have the isometry property

$$E[\mathbf{M}_d(t)\mathbf{M}_d(t)^T] = \int_{\mathbb{R}} f_t(s)\Sigma_L f_t(s)\,ds, \quad t \in \mathbb{R}, \tag{2.18}$$

and we see that the second-order properties of the MFLP $\{\mathbf{M}_d(t)\}_{t\in\mathbb{R}}$ are specified by $E[\mathbf{M}_d(t)] = 0$ and covariance matrices

$$\Gamma(s,t) = E[\mathbf{M}_d(s)\mathbf{M}_d(t)^T] = [\gamma_{ij}(s,t)]_{i,j=1}^m, \quad s, t \in \mathbb{R},$$

where for $s, t \in \mathbb{R}$,

$$\gamma_{ij}(s,t) = E[M_d^i(s) M_d^j(t)] = \operatorname{cov}(L^i(1), L^j(1)) \langle f_t^{ii}, f_s^{jj} \rangle_{L^2(\mathbb{R})}$$
$$= \frac{C \operatorname{cov}(L^i(1), L^j(1))}{2\Gamma(d_i)\Gamma(d_j)} \left[|t|^{d_i+d_j+1} - |t-s|^{d_i+d_j+1} + |s|^{d_i+d_j+1} \right],$$

where f_t^{kk} denotes the kth diagonal element of the matrix function f_t and C is a constant given by

$$C = \frac{1}{d_i + d_j + 1} + \int_0^\infty [(1-u)^{d_i} - u^{d_i}][(1-u)^{d_j} - u^{d_j}]\, du.$$

Recall that $\operatorname{cov}(L^i(1), L^j(1)) = \int_{\mathbb{R}^m} x^i x^j \, v(dx)$, where $x = (x^1, \dots, x^m)^T \in \mathbb{R}^m$. Hence, the MFLP $\{\mathbf{M}_d(t)\}_{t \in \mathbb{R}}$ inherits its dependence structure from the driving Lévy process $\{\mathbf{L}(t)\}_{t \in \mathbb{R}}$.

To the end of this paper we use the notation

$$\Gamma(h) = E[\mathbf{X}(t+h)\mathbf{X}(t)^T] = [\gamma_{ij}(h)]_{i,j=1}^m,$$

if the series $\{\mathbf{X}(t)\}_{t \in \mathbb{R}^m}$ is stationary. We shall refer to $\Gamma(h)$ as the covariance matrix at lag h. Notice that, if $\{\mathbf{X}(t)\}_{t \in \mathbb{R}^m}$ is stationary with covariance matrix function Γ, then for each j, $\{X^j(t)\}_{t \in \mathbb{R}}$, $j = 1, \dots, m$ is stationary with covariance matrix function γ_{jj}. The function γ_{ij}, $i \neq j$, is called the cross-covariance function of the two series $\{X^i(t)\}_{t \in \mathbb{R}}$ and $\{X^j(t)\}_{t \in \mathbb{R}}$. It should be noted that γ_{ij} is not in general the same as γ_{ji}.

The sample path properties of a MFLP are analogous to the one-dimensional case. We therefore omit the proof of the following proposition and refer to [17].

Proposition 2.4 (*Sample path properties*). *Every MFLP is a process with long memory and stationary increments but cannot be self-similar. Moreover, it is symmetric and Hölder continuous of every order less than* $\min(d_1, \dots, d_m)$.

In particular, a MFLP has less smooth sample paths than a fractional Brownian motion. Note also, that the upper bound on the Hölder exponent of the MFLP cannot be improved. In fact, if the Lévy measure v is not finite, the sample paths of MFLPs are not Hölder continuous with probability 1 for every order $\beta > \min(d_1, \dots, d_m)$.

MFLPs are not always semimartingales. The proof of the following theorem is the same as for a univariate FLP. We refer to [17].

Theorem 2.5. (i) *If* $v(\mathbb{R}) < \infty$, *the sample paths of* $\mathbf{M}_d$ *are of finite total variation on compacts and hence*, $\mathbf{M}_d$ *is a semimartingale*.

(ii) *Define for* $0 < \alpha < 2$ *the parameter* $\tilde{H} = (\tilde{H}_1, \dots, \tilde{H}_m)^T$, *where* $\tilde{H}_j = d_j + 1/\alpha$ *such that* $0 < \tilde{H}_j < 1$ *for all* $j = 1, \dots, m$. *Assume that* $v(dx) = g(x)\, dx$, *where* g *is measurable and satisfies* $\|g(x)\| \sim \|x\|^{-1-\alpha}$ *as* $x \to 0$ and $||g(x)|| \leqslant C||x||^{-1-\alpha}$ for all $x \in \mathbf{R}^m$ *Then the sample paths of* $\mathbf{M}_d$ *are a.s. of infinite total variation on compacts*.

Corollary 2.6. *Analogously to the one-dimensional case it can be shown that the quadratic variation of* $\mathbf{M}_d$ *is a.s. zero* (*see* [17]). *Thus, it follows that if* v *is of form* (ii), *the corresponding MFLP cannot be a semimartingale*.

2.4. Integration with respect to MFLPs

As stated in Corollary 2.6, MFLPs are not always semimartingales and thus ordinary Itô integration theory cannot be applied. This section therefore contains the integration theory for stochastic integrals with respect to MFLPs, which is heavily based on the integration theory with respect to a one-dimensional FLP (see [17]).

We define the space H as the completion of $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with respect to the norm

$$\|g\|_H := \left(E[L(1)^2] \int_{\mathbb{R}} (I_-^d g)^2(u)\, du \right)^2, \tag{2.19}$$

where $(I_-^d g)(u) = \frac{1}{\Gamma(d)} \int_u^\infty (s-u)^{d-1} g(s)\, ds,\ \ u \in \mathbb{R}$, is the right-sided Riemann–Liouville fractional integral of order d of the function $g : \mathbb{R} \to \mathbb{R}$, $g \in L^1(\mathbb{R})$. Then for every $g \in H$ it holds a.s.,

$$\int_{\mathbb{R}} g(s)\, M_d(ds) = \int_{\mathbb{R}} (I_-^d g)(u)\, L(du), \tag{2.20}$$

where $\{M_d(t)\}_{t\in\mathbb{R}}$ is a univariate FLP (see [17]).

Now let $G : \mathbb{R} \to M_m(\mathbb{R})$ be a matrix function whose components $G_{jk} : \mathbb{R} \to \mathbb{R}$, $j,k = 1,\dots,m$, are in the space H. To ease notation we write $G \in H_m$. Moreover, let $\{\mathbf{M}_d(t)\}_{t\in\mathbb{R}}$ denote an m-dimensional FLP. Then we define

$$\int_{\mathbb{R}} G(t)\, \mathbf{M}_d(dt) = \begin{pmatrix} \int_{\mathbb{R}} (I_-^{d_1} G_{11})(u)\, L^1(du) + & \cdots & + \int_{\mathbb{R}} (I_-^{d_m} G_{1m})(u)\, L^m(du) \\ \vdots & & \vdots \\ \int_{\mathbb{R}} (I_-^{d_1} G_{m1})(u)\, L^1(du) + & \cdots & + \int_{\mathbb{R}} (I_-^{d_m} G_{mm})(u)\, L^m(du) \end{pmatrix}. \tag{2.21}$$

Denoting the coordinates of $\mathbf{M}_d$ by M_{d_j}, the jth element $(\int G(t)\,\mathbf{M}_d(dt))^j$ of $\int G(t)\,\mathbf{M}_d(dt)$ is given by $\sum_{k=1}^m \int G_{jk}(t)\, M_{d_k}(dt) = \sum_{k=1}^m \int (I_-^{d_k} G_{jk})(t) L^k(dt)$, where the integrals are one-dimensional stochastic integrals as in (2.20) in an L^2-sense, i.e. the integration can be understood component-wise. It is therefore obvious that integral (2.21) is well-defined, whenever $G \in H_m$. This leads to the following isometry property.

Proposition 2.7. *Let $F : \mathbb{R} \to M_m(\mathbb{R})$ and $G : \mathbb{R} \to M_m(\mathbb{R})$ be matrix functions whose components $F_{ij} : \mathbb{R} \to \mathbb{R}$ and $G_{ij} : \mathbb{R} \to \mathbb{R}$, $i, j = 1, \dots, m$ are in the space H. Then*

$$E\left[\left(\int_{\mathbb{R}} F(t)\, \mathbf{M}_d(dt) \right) \left(\int_{\mathbb{R}} G(u)\, \mathbf{M}_d(du) \right)^T \right] = R, \tag{2.22}$$

where R is an $m \times m$-matrix of which the (i, j)-element is given by

$$\begin{aligned} R^{ij} &= E\left[\sum_{k=1}^m \sum_{l=1}^m \int_{\mathbb{R}} F_{ik}(t)\, M_{d_k}(dt) \int_{\mathbb{R}} G_{jl}(u)\, M_{d_l}(du) \right] \\ &= \sum_{k=1}^m \sum_{l=1}^m \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{K \operatorname{cov}(L(k), L(l))}{\Gamma(d_k)\Gamma(d_l)} F_{ik}(t) G_{jl}(u) |t-u|^{d_k+d_l-1}\, dt\, du, \end{aligned}$$

where $K > 0$ is a constant.

Proof. It is a well-known fact that (see e.g. [13, p. 405])

$$\int_{-\infty}^{\min(u,t)} (t-s)^{d_k-1}(u-s)^{d_l-1}\,ds = K\,|t-u|^{d_k+d_l-1}, \quad u,t \in \mathbb{R},$$

where

$$K = \begin{cases} \Gamma(1-d_k-d_l)\Gamma(d_l)/\Gamma(1-d_k), & u<t, \\ \Gamma(1-d_k-d_l)\Gamma(d_k)/\Gamma(1-d_l), & u>t. \end{cases}$$

Hence, by (2.21),

$$\begin{aligned} R^{ij} &= E\left[\sum_{k=1}^{m}\sum_{l=1}^{m}\int_{\mathbb{R}} F_{ik}(t)\,M_{d_k}(dt)\int_{\mathbb{R}} G_{jl}(u)\,M_{d_l}(du)\right] \\ &= \sum_{k=1}^{m}\sum_{l=1}^{m}\frac{\operatorname{cov}(L(k),L(l))}{\Gamma(d_k)\Gamma(d_l)}\int_{\mathbb{R}}\int_s^{\infty}\int_s^{\infty} F_{ik}(t)G_{jl}(u)(t-s)^{d_k-1}(u-s)^{d_l-1}\,dt\,du\,ds \\ &= \sum_{k=1}^{m}\sum_{l=1}^{m}\frac{\operatorname{cov}(L(k),L(l))}{\Gamma(d_k)\Gamma(d_l)}\int_{\mathbb{R}}\int_{\mathbb{R}} F_{ik}(t)G_{jl}(u)\int_{-\infty}^{\min(t,u)}(t-s)^{d_k-1}(u-s)^{d_l-1}\,ds\,dt\,du, \\ &= \sum_{k=1}^{m}\sum_{l=1}^{m}\int_{\mathbb{R}}\int_{\mathbb{R}}\frac{K\operatorname{cov}(L(k),L(l))}{\Gamma(d_k)\Gamma(d_l)}F_{ik}(t)G_{jl}(u)|t-u|^{d_k+d_l-1}\,dt\,du, \end{aligned}$$

where we have used Fubini's theorem. □

2.5. The spectral representation of MFLPs

In [18] it is shown that for every m-dimensional Lévy process $\mathbf{L} = \{\mathbf{L}(t)\}_{t\in\mathbb{R}}$ with $E[\mathbf{L}(1)] = 0$ and $E[\mathbf{L}(1)\mathbf{L}(1)^T] = \Sigma_L < \infty$ there exists an m-dimensional orthogonal random measure Φ_L such that $E[\Phi_L(\Delta)] = 0$ and $E[\Phi_L(\Delta)\Phi_L(\Delta)^*] = \frac{1}{2\pi}\Sigma_L\Lambda(\Delta)$ for any bounded Borel set Δ, where Λ denotes the Lebesgue measure. The random measure Φ_L is uniquely determined by

$$\Phi_L([a,b)) = \int_{\mathbb{R}} \frac{e^{-i\lambda a} - e^{-i\lambda b}}{2\pi i\lambda}\,\mathbf{L}(d\lambda) \tag{2.23}$$

for all $-\infty < a < b < \infty$. Moreover,

$$\mathbf{L}(t) = \int_{\mathbb{R}} \frac{e^{i\lambda t}-1}{i\lambda}\,\Phi_L(d\lambda), \quad t\in\mathbb{R}. \tag{2.24}$$

Finally, for any function $f \in L^2(M_m(\mathbb{C}))$,

$$\int_{\mathbb{R}} f(\lambda)\,\Phi_L(d\lambda) = \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\lambda t} f(\lambda)\,d\lambda\,\mathbf{L}(dt) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\hat{f}(t)\,\mathbf{L}(dt), \tag{2.25}$$

$$\int_{\mathbb{R}}\hat{f}(t)\,\mathbf{L}(dt) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda t}\hat{f}(t)\,dt\,\Phi_L(d\lambda) = \sqrt{2\pi}\int_{\mathbb{R}} f(\lambda)\,\Phi_L(d\lambda). \tag{2.26}$$

Here,

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\lambda t} f(\lambda)\,d\lambda \quad \text{and} \quad f(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{i\lambda t}\hat{f}(t)\,dt$$

are the Fourier transform and the inverse Fourier transform, respectively. We will use these results to obtain a spectral representation for MFLPs and integrals with respect to them.

Theorem 2.8. *Let* $\mathbf{M}_d = \{\mathbf{M}_d(t)\}_{t\in\mathbb{R}}$ *be an m-dimensional FLP. Then* $\mathbf{M}_d$ *has the spectral representation*

$$\mathbf{M}_d(t) = \int_{\mathbb{R}} (e^{i\lambda t} - 1) C(i\lambda)\, \Phi_L(d\lambda), \quad t \in \mathbb{R}, \tag{2.27}$$

where Φ_L *is the random orthogonal measure defined in* (2.23) *and*

$$C(i\lambda) = \begin{pmatrix} (i\lambda)^{-d_1-1} & & 0 \\ & \ddots & \\ 0 & & (i\lambda)^{-d_m-1} \end{pmatrix}.$$

Furthermore, let

$$\Phi_M([a,b]) = \int_{\mathbb{R}} I_{(a,b)}(\lambda) D(i\lambda)\, \Phi_L(d\lambda), \quad a < b, \tag{2.28}$$

define a random measure, where

$$D(i\lambda) = \begin{pmatrix} (i\lambda)^{-d_1} & & 0 \\ & \ddots & \\ 0 & & (i\lambda)^{-d_m} \end{pmatrix}.$$

Then

$$\Phi_M([a,b]) = \int_{\mathbb{R}} \frac{e^{-ias} - e^{-ibs}}{2\pi i s}\, \mathbf{M}_d(ds). \tag{2.29}$$

Proof. Observe that [10, Formula 4, p. 1081]

$$\frac{1}{\Gamma(d+1)} \int_{\mathbb{R}} [(b-s)_+^d - (a-s)_+^d] e^{i\lambda s}\, ds = \frac{e^{i\lambda b} - e^{i\lambda a}}{(i\lambda)^{d+1}}. \tag{2.30}$$

Using (2.26) and (2.30) we obtain

$$\begin{aligned} &\mathbf{M}_d(b) - \mathbf{M}_d(a) \\ &\quad = \int_{\mathbb{R}} [f_b(s) - f_a(s)]\, \mathbf{L}(ds) \\ &\quad = \int_{\mathbb{R}}\int_{\mathbb{R}} \begin{pmatrix} \frac{1}{\Gamma(d_1+1)}[(b-s)_+^{d_1} - (a-s)_+^{d_1}] & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\Gamma(d_m+1)}[(b-s)_+^{d_m} - (a-s)_+^{d_m}] \end{pmatrix} \\ &\qquad \times e^{i\lambda s}\, ds\, \Phi_L(d\lambda) \\ &\quad = \int_{\mathbb{R}} (e^{i\lambda b} - e^{i\lambda a}) C(i\lambda)\, \Phi_L(d\lambda). \end{aligned}$$

It remains to prove (2.29):

$$
\begin{aligned}
&\int_{\mathbb{R}} \frac{e^{-ias}-e^{-ibs}}{2\pi i s}\,\mathbf{M}_d(ds)\\
&=\begin{pmatrix}
\int_{\mathbb{R}} \frac{1}{\Gamma(d_1)} \int_u^{\infty} (s-u)^{d_1-1} \frac{e^{-ias}-e^{-ibs}}{2\pi i s}\, ds\, L^1(du)\\
\vdots\\
\int_{\mathbb{R}} \frac{1}{\Gamma(d_m)} \int_u^{\infty} (s-u)^{d_m-1} \frac{e^{-ias}-e^{-ibs}}{2\pi i s}\, ds\, L^m(du)
\end{pmatrix}\\
&=\begin{pmatrix}
\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{e^{iu\lambda}(s-u)_+^{d_1-1}}{\Gamma(d_1)} \frac{e^{-ias}-e^{-ibs}}{2\pi i s}\, ds\, du\, \Phi_L^1(d\lambda)\\
\vdots\\
\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{e^{iu\lambda}(s-u)_+^{d_m-1}}{\Gamma(d_m)} \frac{e^{-ias}-e^{-ibs}}{2\pi i s}\, ds\, du\, \Phi_L^m(d\lambda)
\end{pmatrix}\\
&=\begin{pmatrix}
\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{e^{iu\lambda}(s-u)_+^{d_1-1}}{\Gamma(d_1)}\, du \frac{e^{-ias}-e^{-ibs}}{2\pi i s}\, ds\, \Phi_L^1(d\lambda)\\
\vdots\\
\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{e^{iu\lambda}(s-u)_+^{d_m-1}}{\Gamma(d_m)}\, du \frac{e^{-ias}-e^{-ibs}}{2\pi i s}\, ds\, \Phi_L^m(d\lambda)
\end{pmatrix}\\
&=\int_{\mathbb{R}}\int_{\mathbb{R}} D(i\lambda) e^{i\lambda s} \frac{e^{-ias}-e^{-ibs}}{2\pi i s}\, ds\, \Phi_L(d\lambda)\\
&=\int_{\mathbb{R}} D(i\lambda) I_{(a,b)}(\lambda)\, \Phi_L(d\lambda) = \Phi_M([a,b]). \qquad \square
\end{aligned}
$$

Remark 2.9. A similar calculation as in the proof of Theorem 2.8 yields

$$\int_{\mathbb{R}} g(t)\, \mathbf{M}_d(dt) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda t} g(t) D(i\lambda)\, dt\, \Phi_L(d\lambda) = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\lambda t} g(t)\, dt\, \Phi_M(d\lambda). \quad (2.31)$$

2.6. Multivariate CARMA processes

Our aim in this paper is to define a multivariate FICARMA process. Univariate FICARMA processes are closely related to univariate CARMA processes as they are obtained via a fractional integration of the CARMA kernel (see [9]). Recently, using a state space representation and the spectral representation (2.24) of the driving Lévy process, a multivariate Lévy-driven CARMA model of order (p, q), $q < p$ was introduced in [18]. We give a brief review of the multivariate CARMA (MCARMA) processes, where we focus on causal MCARMA processes.

Definition 2.10 (*MCARMA process*). Let $\mathbf{L} = \{\mathbf{L}(t)\}_{t\in\mathbb{R}}$ be a two-sided square integrable m-dimensional Lévy-process with $E[\mathbf{L}(1)\mathbf{L}(1)^T] = \Sigma_L < \infty$. An m-dimensional causal Lévy-driven continuous time autoregressive moving average process $\{\mathbf{Y}(t)\}_{t\in\mathbb{R}}$ of order (p, q), $p > q$ (MCARMA(p, q) process) is defined to be the stochastic process having the spectral representation

$$\mathbf{Y}(t) = \int_{-\infty}^{\infty} e^{i\lambda t} P(i\lambda)^{-1} Q(i\lambda)\, \Phi_L(d\lambda), \quad t \in \mathbb{R}, \quad (2.32)$$

where Φ_L is the Lévy orthogonal random measure in (2.23) satisfying $E[\Phi_L(d\lambda)] = 0$ and $E[\Phi_L(d\lambda)\Phi(d\lambda)^*] = \frac{1}{2\pi}\Sigma_L d\lambda$. Here,

$$P(z) := I_m z^p + A_1 z^{p-1} + \cdots + A_p, \tag{2.33}$$

$$Q(z) := B_0 z^q + B_1 z^{q-1} + \cdots + B_q, \tag{2.34}$$

where $A_j \in M_m(\mathbb{R})$, $j = 1, \dots, p$ and $B_j \in M_m(\mathbb{R})$ are matrices satisfying $A_p \neq 0$, $B_q \neq 0$ and $\mathcal{N}(P) := \{z \in \mathbb{C} : det(P(z)) = 0\} \subset (-\infty, 0) + i\mathbb{R}$.

The name "multivariate continuous time ARMA process" is indeed appropriate, since an MCARMA process $\mathbf{Y}$ can be interpreted as a solution to the pth order m-dimensional differential equation

$$P(D)\mathbf{Y}(t) = Q(D)D\mathbf{L}(t),$$

where D denotes the differentiation operator. Moreover, the spectral representation (2.32) is the continuous time analogue of the spectral representation of multivariate discrete time ARMA processes (see e.g. [8, Section 11.8]).

The following Proposition 2.11 shows that for $m = 1$ the well-known univariate CARMA processes are obtained. In fact, like univariate CARMA processes, MCARMA processes allow for a short memory moving average representation.

Proposition 2.11. *The MCARMA process* (2.32) *can be represented as a causal moving average process*

$$\mathbf{Y}(t) = \int_{\mathbb{R}} g(t-s)\,\mathbf{L}(ds), \quad t \in \mathbb{R}, \tag{2.35}$$

where the kernel matrix function $g : \mathbb{R} \to M_m(\mathbb{R})$ *is given by*

$$g(t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\mu t} P(i\mu)^{-1} Q(i\mu)\, d\mu, \quad t \in \mathbb{R}, \tag{2.36}$$

and satisfies $g(t) = 0$ *for* $t < 0$.

We finally summarize the second order properties.

Proposition 2.12. *Let* $\mathbf{Y} = \{\mathbf{Y}(t)\}_{t\in\mathbb{R}}$ *be the MCARMA process defined by* (2.32). *Then its covariance matrix function is given by*

$$\Gamma_Y(h) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda h} P(i\lambda)^{-1} Q(i\lambda)\Sigma_L Q(i\lambda)^*(P(i\lambda)^{-1})^*\, d\lambda, \quad h \in \mathbb{R}.$$

and its spectral density has the form

$$f_Y(\lambda) = \frac{1}{2\pi} P(i\lambda)^{-1} Q(i\lambda)\Sigma_L Q(i\lambda)^*(P(i\lambda)^{-1})^*, \quad \lambda \in \mathbb{R}. \tag{2.37}$$

Remark 2.13. MCARMA processes belong to the class of short memory moving average processes. In the next section we define multivariate fractionally integrated CARMA (MFICARMA) processes which show long memory properties.

3. Multivariate fractionally integrated CARMA processes

In this section we develop multivariate fractionally integrated CARMA (FICARMA) processes which exhibit long range dependence. So far only univariate FICARMA processes have been defined and investigated (see [7,9]). We first give a meaning to "long memory".

Definition 3.1 (*Long memory process*). Let $\mathbf{X} = \{\mathbf{X}_t\}_{t\in\mathbb{R}}$ be a stationary stochastic process and $\Gamma_X(h) = \operatorname{cov}(\mathbf{X}_{t+h}, \mathbf{X}_t)$, $h \in \mathbb{R}$, be its autocovariance function. If there exist $d = (d_1, \ldots, d_m)^T$ such that $0 < d_j < 0.5$ for all $j = 1, \ldots, m$ and constants $c_\Gamma^{ij} > 0$, $i, j = 1, \ldots, m$, such that

$$\lim_{h\to\infty} \Gamma_X^{ij}(h) = c_\Gamma^{ij} |h|^{2\tilde{d}-1} \quad \text{for all } i, j = 1, \ldots, m, \tag{3.1}$$

where Γ_X^{ij} denotes the (i, j)th element of the matrix Γ_X and $\tilde{d} = \max(d_1, \ldots, d_m) \in (0, 0.5)$. Then $\mathbf{X}$ is a stationary process with long memory.

3.1. Representations of MFICARMA processes

In one dimension, starting from a short memory moving average process, there are at least two possible ways to construct a long memory moving average process:

(I) a fractional integration of the kernel of the short memory process,
(II) a substitution of the driving Lévy process by the corresponding FLP.

Both approaches lead to the same long memory L^2-process (see [17]). For the univariate CARMA process (I) leads to the univariate FICARMA(p, d, q) process (see [9])

$$Y_d(t) = \int_{-\infty}^{t} g_d(t-s)\, L(ds), \quad t \in \mathbb{R}, \tag{3.2}$$

where the driving Lévy process $L = \{L(t)\}_{t\in\mathbb{R}}$ satisfies $E[L(1)] = 0$ and $E[L(1)^2] < \infty$ and the kernel

$$g_d(t) = (I_+^d g)(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{it\mu} (i\mu)^{-d} \frac{q(i\mu)}{p(i\mu)}\, d\mu, \quad t \in \mathbb{R}, \tag{3.3}$$

is the left-sided Riemann–Liouville fractional integral of order d of the univariate CARMA kernel g. Here $0 < d < 0.5$ is referred to as the fractional integration parameter and

$$p(z) := z^p + a_1 z^{p-1} + \cdots + a_p \quad \text{and} \quad q(z) := b_0 z^q + b_1 z^{q-1} + \cdots + b_q,$$

where $a_p \neq 0$, $b_q \neq 0$. The polynomials $p(\cdot)$ and $q(\cdot)$ are referred to as the autoregressive and moving average polynomial, respectively.

We apply approach (I) to MCARMA processes to obtain MFICARMA processes, i.e. we fractionally integrate the MCARMA kernel g as given in (2.36) (observe that $g \in H_m$) and obtain for $t \in \mathbb{R}$,

$$
\begin{aligned}
g_d(t) &:= \begin{bmatrix} (I_+^{d_1} g_{11})(u) & \dots & (I_+^{d_m} g_{1m})(u) \\ \vdots & & \vdots \\ (I_+^{d_1} g_{m1})(u) & \dots & (I_+^{d_m} g_{mm})(u) \end{bmatrix} \\
&= \int_0^t \begin{bmatrix} g_{11}(t-u) & \dots & g_{1m}(t-u) \\ \vdots & & \vdots \\ g_{m1}(t-u) & \dots & g_{mm}(t-u) \end{bmatrix} \begin{bmatrix} u^{d_1-1}/\Gamma(d_1) & & 0 \\ & \ddots & \\ 0 & & u^{d_m-1}/\Gamma(d_m) \end{bmatrix} du \\
&= \frac{1}{2\pi} \int_0^t \int_{\mathbb{R}} e^{i\mu(t-u)} P(i\mu)^{-1} Q(i\mu)\, d\mu \begin{bmatrix} u^{d_1-1}/\Gamma(d_1) & & 0 \\ & \ddots & \\ 0 & & u^{d_m-1}/\Gamma(d_m) \end{bmatrix} du \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\mu t} P(i\mu)^{-1} Q(i\mu) \int_0^t e^{-i\mu u} \begin{bmatrix} u^{d_1-1}/\Gamma(d_1) & & 0 \\ & \ddots & \\ 0 & & u^{d_m-1}/\Gamma(d_m) \end{bmatrix} du\, d\mu \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\mu t} P(i\mu)^{-1} Q(i\mu) \begin{bmatrix} (i\mu)^{-d_1} & & 0 \\ & \ddots & \\ 0 & & (i\mu)^{-d_m} \end{bmatrix} d\mu \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\mu t} P(i\mu)^{-1} Q(i\mu) D(i\mu)\, d\mu. \qquad (3.4)
\end{aligned}
$$

Note that $g_d(t) = 0$ for all $t \leqslant 0$ and $g_d \in L^2(M_m(\mathbb{R}))$. Moreover, for $m = 1$, (3.4) is equivalent to (3.3). This leads to the following definition.

Definition 3.2 (*MFICARMA process I*). For $d = (d_1, \dots, d_m)^T$ such that $0 < d_j < 0.5$, $j = 1, \dots, m$ and for $p > q$ the multivariate fractionally integrated CARMA(p, d, q) (MFICARMA) process driven by the m-dimensional Lévy process $\mathbf{L} = \{\mathbf{L}(t)\}_{t \in \mathbb{R}}$ with $E[\mathbf{L}(1)] = 0$ and $E[\mathbf{L}(1)\mathbf{L}(1)^T] = \Sigma_L < \infty$ is defined by

$$\mathbf{Y}_d(t) = \int_{-\infty}^t g_d(t-s)\, \mathbf{L}(ds), \quad t \in \mathbb{R}, \qquad (3.5)$$

where the fractionally integrated kernel g_d is given as in (3.4) and where the polynomials $P(\cdot)$ and $Q(\cdot)$ are defined as in (2.33) and (2.34), respectively.

Remark 3.3. We show in Theorem 3.8 below that the MFICARMA process $\mathbf{Y}_d$ in (3.5) is indeed well-defined.

Now, we turn our attention to approach (II) and substitute in the MCARMA representation the driving Lévy process by the corresponding MFLP.

Definition 3.4 (*MFICARMA process II*). For $d = (d_1, \ldots, d_m)^T$ such that $0 < d_j < 0.5$, $j = 1, \ldots, m$ and for $p > q$ the multivariate fractionally integrated CARMA(p, d, q) (MFICARMA) process driven by the m-dimensional FLP $\mathbf{M}_d = \{\mathbf{M}_d(t)\}_{t\in\mathbb{R}}$ is defined by

$$\mathbf{Y}_d(t) = \int_{-\infty}^{t} g(t-s)\,\mathbf{M}_d(ds), \quad t \in \mathbb{R}, \tag{3.6}$$

where the kernel g is the CARMA kernel given in (2.36).

Representation (3.6) is equal to (3.5). In fact, using (2.21), we have

$$\begin{aligned}
&\int_{\mathbb{R}} g(t-s)\,\mathbf{M}_d(ds) \\
&= \begin{bmatrix} \int_{\mathbb{R}} g_{11}(t-s)\,M^1_{d_1}(ds) + \cdots + \int_{\mathbb{R}} g_{1m}(t-s)\,M^m_{d_m}(ds) \\ \vdots \\ \int_{\mathbb{R}} g_{m1}(t-s)\,M^1_{d_1}(ds) + \cdots + \int_{\mathbb{R}} g_{mm}(t-s)\,M^m_{d_m}(ds) \end{bmatrix} \\
&= \begin{bmatrix} \int_{\mathbb{R}} \left(\frac{1}{\Gamma(d_1)} \int_u^\infty (s-u)^{d_1-1} g_{11}(t-s)\,ds\right) L^1(du) + \cdots + \int_{\mathbb{R}} \left(\frac{1}{\Gamma(d_m)} \int_u^\infty (s-u)^{d_m-1} g_{1m}(t-s)\,ds\right) L^m(du) \\ \vdots \\ \int_{\mathbb{R}} \left(\frac{1}{\Gamma(d_1)} \int_u^\infty (s-u)^{d_1-1} g_{m1}(t-s)\,ds\right) L^1(du) + \cdots + \int_{\mathbb{R}} \left(\frac{1}{\Gamma(d_m)} \int_u^\infty (s-u)^{d_m-1} g_{mm}(t-s)\,ds\right) L^m(du) \end{bmatrix} \\
&= \int_{\mathbb{R}} \begin{bmatrix} \frac{1}{\Gamma(d_1)} \int_0^\infty s^{d_1-1} g_{11}(t-s-u)\,ds & \ldots & \frac{1}{\Gamma(d_m)} \int_0^\infty s^{d_m-1} g_{1m}(t-s-u)\,ds \\ \vdots & & \vdots \\ \frac{1}{\Gamma(d_1)} \int_0^\infty s^{d_1-1} g_{m1}(t-s-u)\,ds & \ldots & \frac{1}{\Gamma(d_m)} \int_0^\infty s^{d_m-1} g_{mm}(t-s-u)\,ds \end{bmatrix} \mathbf{L}(du) \\
&= \int_{\mathbb{R}} \begin{bmatrix} (I^{d_1}_+ g_{11})(u) & \ldots & (I^{d_m}_+ g_{1m})(u) \\ \vdots & & \vdots \\ (I^{d_1}_+ g_{m1})(u) & \ldots & (I^{d_m}_+ g_{mm})(u) \end{bmatrix} \mathbf{L}(du) = \int_{\mathbb{R}} g_d(t-s)\,\mathbf{L}(du).
\end{aligned}$$

As we will see in Section 3.2 representation (3.5) is useful to obtain distributional and sample path properties, whereas representation (3.6) is useful for simulations and essential to obtain a spectral representation for MFICARMA processes.

Theorem 3.5. *The MFICARMA(p, d, q) process $\mathbf{Y}_d = \{\mathbf{Y}_d(t)\}_{t\in\mathbb{R}}$ has the spectral representation*

$$\begin{aligned}
\mathbf{Y}_d(t) &= \int_{\mathbb{R}} e^{i\mu t} P(i\mu)^{-1} Q(i\mu) D(i\mu)\,\Phi_L(d\mu) \\
&= \int_{\mathbb{R}} e^{i\mu t} P(i\mu)^{-1} Q(i\mu)\,\Phi_M(d\mu), \quad t \in \mathbb{R},
\end{aligned} \tag{3.7}$$

where Φ_L is the random orthogonal measure corresponding to the Lévy process $\mathbf{L}$ and Φ_M is the random measure defined in Theorem 2.8.

Proof. We use equality (2.31) and obtain

$$\begin{aligned}\int_{\mathbb{R}} g(t-s)\,\mathbf{M}_d(ds) &= \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{2\pi} e^{i\mu(t-s)} P(i\mu)^{-1} Q(i\mu)\, d\mu\, \mathbf{M}_d(ds)\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{2\pi} e^{i\mu(t-s)} e^{i\lambda s} P(i\mu)^{-1} Q(i\mu) D(i\lambda)\, d\mu\, ds\, \Phi_L(d\lambda)\\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{i\mu t} P(i\mu)^{-1} Q(i\mu) D(i\lambda) \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(\lambda-\mu)s}\, ds\, d\mu\, \Phi_L(d\lambda)\\ &= \int_{\mathbb{R}} e^{i\mu t} P(i\mu)^{-1} Q(i\mu) D(i\mu)\, \Phi_L(d\mu)\\ &= \int_{\mathbb{R}} e^{i\mu t} P(i\mu)^{-1} Q(i\mu)\, \Phi_M(d\mu). \qquad \square\end{aligned}$$

Remark 3.6. Note that for $d = 0$ the MCARMA processes are obtained. Moreover, an MFICARMA process $\mathbf{Y}_d$ can be interpreted as a solution to the pth order m-dimensional formal differential equation

$$P(D)\mathbf{Y}_d(t) = Q(D)D\mathbf{M}_d(t),$$

where D denotes the differentiation operator. This shows that MFICARMA processes are the continuous-time analogue of the well-known discrete time multivariate fractionally integrated ARMA (ARFIMA) processes (see e.g. [8]).

Thus, the so-called embedding problem arises.

Definition 3.7. A discrete time process $\{X(n)\}_{n\in\mathbb{Z}}$ is said to be embedded in a continuous time process $\{Y(t)\}_{t\in\mathbb{R}}$ if the continuous time process sampled at integer times $\{Y(n)\}_{n\in\mathbb{Z}}$ has the same autocorrelation function as the process $\{X(n)\}_{n\in\mathbb{Z}}$.

In general the question of whether or not there exists a CARMA process Y whose autocovariance function at integer times coincides with that of a given ARMA process is referred to as the embedding problem. It is well-known that every CARMA autocovariance function when restricted to the integers is an ARMA autocovariance function. Thus the embedding problem is equivalent to the question of whether the class of discrete time ARMA autocovariance functions is the same as the class of CARMA autocovariance functions restricted to the integers. Ref. [3] answers this question in the negative by showing that an ARMA(p, q) process with unit root cannot be embedded in any CARMA process. In particular, the problem of finding a simple characterization of the discrete time ARMA processes which are embeddable remains open. Furthermore, the embedding problem is closely connected with the problem of the identification of a CARMA process from observations at integer times. However, [4] gives examples of AR(2) processes that can be embedded in CARMA$(2, 1)$ as well as in CARMA$(4, 2)$ processes. Hence, based only on observations at integer times it will not be possible to distinguish between these CARMA processes.

In the fractionally integrated case the embedding and identification problems are even more complicated. Ref. [9] makes a comparison of the autocorrelation functions at integer times of the

FICARMA$(1, d, 0)$ process and the process obtained by fractionally integrating (in the discrete time sense) the ARMA process obtained by sampling the CARMA process at integer times. The result is that in the fractionally integrated case the autocorrelations do *not* coincide, though the behavior of the autocorrelation functions is quite similar. Therefore the question under which conditions a stationary (univariate or multivariate) discrete time long memory process can be represented as a discretely sampled FICARMA process is still open and has to be delayed to future work.

3.2. Properties of MFICARMA processes

Having defined MFICARMA processes, we consider their distributional, second-order and sample path properties. First note that, since (3.5) is a moving average process, the MFICARMA process is stationary [1, Theorem 4.3.16].

Theorem 3.8 (*Infinite divisibility*). *The MFICARMA process as given in* (3.5) *is well-defined in* $L^2(\Omega, P)$. *For all* $t \in \mathbb{R}$ *the distribution of* $\mathbf{Y}_d(t)$ *is infinitely divisible with characteristic triplet* $(\gamma_Y^t, 0, v_Y^t)$, *where*

$$\gamma_Y^t = -\int_{\mathbb{R}}\int_{\mathbb{R}^m} x g_d(t-s) I_{\{\|g_d(t-s)x\|>1\}}\, v(dx)\, ds \quad \text{and} \tag{3.8}$$

$$v_Y^t(B) = \int_{\mathbb{R}}\int_{\mathbb{R}^m} I_B(g_d(t-s)x)\, v(dx)\, ds, \qquad B \in \mathscr{B}(\mathbb{R}^m) \tag{3.9}$$

and $(\gamma, 0, v)$ *is the characteristic triplet of the driving Lévy process* $\mathbf{L}$.

Proof. Obviously, (3.5) is well-defined in $L^2(\Omega, P)$, since $g_d \in L^2(M_m(\mathbb{R}))$ and (3.8) and (3.9) follow from (2.11). □

Remark 3.9. From Theorem 3.8 follows that the generating triplet of the stationary limiting distribution of $\mathbf{Y}_d(t)$ as $t \to \infty$ is given by $(\gamma_Y^\infty, 0, v_Y^\infty)$, where

$$\gamma_Y^\infty = -\int_0^\infty \int_{\mathbb{R}} x g_d(s) I_{\{\|g_d(s)x\|>1\}}\, v(dx)\, ds \quad \text{and} \tag{3.10}$$

$$v_Y^\infty(B) = \int_0^\infty \int_{\mathbb{R}^m} I_B(g_d(s)x)\, v(dx)\, ds, \qquad B \in \mathscr{B}(\mathbb{R}^m). \tag{3.11}$$

Moreover, if $g_d \in L^r(M_m(\mathbb{R}))$ and the driving Lévy process $\mathbf{L}$ is in $L^r(\Omega, P)$ for some $r > 0$, then the MFICARMA process $\mathbf{Y}_d$ is in $L^r(\Omega, P)$. This follows from the general fact that an infinitely divisible distribution with characteristic triplet (γ, σ, v) has finite r-th moment, if and only if $\int_{\|x\|>\varepsilon} \|x\|^r\, v(dx) < \infty$ for some $\varepsilon > 0$ [24, Corollary 25.8.].

Since the characteristic function of $\mathbf{Y}_d(t)$ for each $t \geqslant 0$ is explicitly given in terms of (3.8) and (3.9), we can investigate the existence of a C_b^∞ density, where C_b^∞ denotes the space of bounded continuous, infinitely often differentiable functions whose derivatives are bounded.

Proposition 3.10. *Suppose that there exist an* $\alpha \in (0, 2)$ *and a constant* $C > 0$ *such that*

$$\int_{\mathbb{R}}\int_{\mathbb{R}^m} |\langle w, g_d(t-s)x\rangle|^2\, I_{\{|\langle w, g_d(t-s)x\rangle| \leqslant 1\}}\, v(dx)\, ds \geqslant C\|w\|^{2-\alpha} \tag{3.12}$$

for any vector w *such that* $\|w\| \geqslant 1$. *Then* $\mathbf{Y}_d(t)$ *has a* C_b^∞ *density.*

Proof. It is sufficient to show that $\int \|w\|^k \|\Phi(w)\| \, dw < \infty$ for any non-negative integer k, where Φ denotes the characteristic function of $\mathbf{Y}_d(t)$ (see e.g. [20, Proposition 0.2]).

The characteristic function of $\mathbf{Y}_d(t)$ is given by

$$\Phi(w) = \exp\left\{\int_{\mathbb{R}} \int_{\mathbb{R}^m} \left[e^{i\langle w, g_d(t-s)x\rangle} - 1 - i\langle w, g_d(t-s)x\rangle\right] \nu(dx)\, ds\right\}. \tag{3.13}$$

Thus,

$$\begin{aligned}\|\Phi(w)\| &= \left(\exp\left\{\int_{\mathbb{R}} \int_{\mathbb{R}^m} \left[e^{i\langle w, g_d(t-s)x\rangle} + e^{-i\langle w, g_d(t-s)x\rangle} - 2\right] \nu(dx)\, ds\right\}\right)^{1/2} \\ &= \exp\left\{\int_{\mathbb{R}} \int_{\mathbb{R}^m} (\cos\langle w, g_d(t-s)x\rangle - 1)\ \nu(dx)\, ds\right\}.\end{aligned}$$

Then, using the inequality $1 - \cos(x) \geqslant 2(x/\pi)^2$ for $\|x\| \leqslant \pi$ and assumption (3.12) we have

$$\begin{aligned}\|\Phi(w)\| &\leqslant \exp\left\{-\tilde{C} \int_{\mathbb{R}} \int_{\mathbb{R}^m} |\langle w, g_d(t-s)x\rangle|^2 I_{\{|\langle w, g_d(t-s)x\rangle| \leqslant 1\}}\, \nu(dx)\, ds\right\} \\ &\leqslant \exp\{-C\|w\|^{2-\alpha}\},\end{aligned}$$

and the proof is complete. $\square$

So far we only used representation (3.5) to derive probabilistic properties. However, having the spectral representation (3.7), we can immediately conclude that the spectral density of an MFICARMA(p, d, q) process has the form

$$f_{Y_d}(\lambda) = \frac{1}{2\pi} P(i\lambda)^{-1} Q(i\lambda) D(i\lambda) \Sigma_L D(i\lambda) Q(i\lambda)^* (P(i\lambda)^{-1})^*, \quad \lambda \in \mathbb{R}.$$

The following proposition is therefore obvious.

Proposition 3.11. *Let $\mathbf{Y}_d(t) = \{\mathbf{Y}_d(t)\}_{t\in\mathbb{R}}$ be an MFICARMA(p, d, q) process. Then it has the covariance matrix function*

$$\Gamma_{Y_d}(h) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\lambda h} P(i\lambda)^{-1} Q(i\lambda) D(i\lambda) \Sigma_L D(i\lambda) Q(i\lambda)^* (P(i\lambda)^{-1})^*\, d\lambda, \quad h \in \mathbb{R}.$$

Alternatively, we can use (2.22) together with representation (3.6) and obtain for $h \geqslant 0$ for the covariance matrix function of an MFICARMA process

$$\begin{aligned}\Gamma^{ij}_{Y_d}(h) &= E\left[\sum_{k=1}^m \sum_{l=1}^m \int_{-\infty}^{t+h} g_{ik}(t+h-s)\, M_{d_k}(ds) \int_{-\infty}^{t} g_{jl}(t-u)\, M_{d_l}(du)\right] \\ &= \sum_{k=1}^m \sum_{l=1}^m \int_{-\infty}^{t+h} \int_{-\infty}^{t} \frac{K \operatorname{cov}(L(k), L(l))}{\Gamma(d_k)\Gamma(d_l)} g_{ik}(t+h-s) g_{jl}(t-u) |s-u|^{d_k+d_l-1}\, ds\, du\end{aligned}$$

and $\Gamma^{ij}_{Y_d}(h) = \Gamma^{ij}_{Y_d}(-h),\ \ h < 0$. It follows

$$\Gamma^{ij}_{Y_d}(h) \sim \sum_{k=1}^m \sum_{l=1}^m |h|^{d_k+d_j-1} \frac{K \operatorname{cov}(L(k), L(l))}{\Gamma(d_k)\Gamma(d_l)} \int_0^\infty \int_0^\infty g_{ik}(s) g_{jl}(u)\, ds\, du \quad \text{as } h \to \infty.$$

Therefore an MFICARMA(p, d, q) process is a ***long memory process*** according to Definition 3.1.

We conclude the analysis of MFICARMA processes with a result on their sample path behavior.

Proposition 3.12 (*Continuity*). *If $g_d \in C_b^1(\mathbb{R})$, then the MFICARMA process $\mathbf{Y}_d$ has a continuous version on every bounded interval I of $\mathbb{R}$.*

Proof. Applying [16, Theorem 2.5], we obtain that $\mathbf{Y}_d$ has a continuous version on $I \subset \mathbb{R}$, if $g_d(0) = 0$ and if for some $\varepsilon > 0$,

$$\sup_{u,v \in I} \left(\log \frac{1}{|u-v|} \right)^{1/2+\varepsilon} \|g_d(u) - g_d(v)\| < \infty.$$

We have $\|g_d(u) - g_d(v)\| \leqslant \|Dg_d(\xi)\| |u-v| \leqslant C|u-v|,\ u \leqslant \xi \leqslant v,\ \xi \in I$. Therefore,

$$\sup_{u,v \in I} \left(\log \frac{1}{|u-v|} \right)^{1/2+\varepsilon} \|g_d(u) - g_d(v)\| \leqslant \sup_{t \in I'} C|t|(-\log |t|)^{1/2+\varepsilon} = \sup_{t \in I'} m(t),$$

where $m(t) = C|t|(-\log |t|)^{1/2+\varepsilon} \leqslant C|t|(-\log |t|) \to 0$ as $t \to 0^+$. Moreover, m is continuous and assumes its maximum on any compact interval. Hence, $\sup_{t \in I'} m(t) < \infty$. □

4. The fractional Ornstein–Uhlenbeck process

Lévy-driven processes of Ornstein–Uhlenbeck (OU) type have been extensively studied over the last years and widely used in applications, especially in the context of finance and econometrics. Several examples of univariate non-Gaussian OU processes can be found in [2], where OU processes are used to model stochastic volatility. Recently multidimensional non-Gaussian OU processes have been considered in [19]. Moreover, [11] discussed among other processes univariate fractional OU processes which were driven by a fractional Brownian motion. In this section we generalize the latter ideas to obtain a multivariate fractional OU process which shows long memory.

Definition 4.1 (*Fractional Ornstein–Uhlenbeck process*). Let $A \in M_m(\mathbb{R})$ be a matrix such that all the eigenvalues of A have negative real part. Let $B \in M_m(\mathbb{R})$ be positive definite and $\mathbf{M}_d = \{\mathbf{M}_d(t)\}_{t \in \mathbb{R}}$ be an m-dimensional FLP as defined in (2.13). We define the fractional Ornstein–Uhlenbeck process by

$$\mathbf{O}_t^{d,A,B} = \int_{-\infty}^{t} e^{A(t-s)} B\, \mathbf{M}_d(ds), \quad t \in \mathbb{R}. \tag{4.1}$$

Remark 4.2. Obviously (4.1) is an MFICARMA$(1, d, 0)$ process and is therefore stationary and well-defined. Moreover, it is a process with long memory.

Without serious loss of generality we assume that the matrix A is diagonalizable. Therefore, let $U \in M_m(\mathbb{R})$ be such that $A = UDU^{-1}$, where $D = \mathrm{diag}(\lambda_i)_{i=1,\ldots,m}$ and λ_i, $i = 1, \ldots, m$, are the eigenvalues of A. Then, when calculating the left-sided Riemann–Liouville fractional integral of the kernel $G(t-s) = e^{A(t-s)}BI_{(0,\infty)}(t-s)$, we obtain

$$G_d(t) := \int_0^\infty e^{A(t-s)}BI_{(0,\infty)}(t-s) \begin{bmatrix} \frac{s^{d_1-1}}{\Gamma(d_1)} & & 0 \\ & \ddots & \\ 0 & & \frac{s^{d_m-1}}{\Gamma(d_m)} \end{bmatrix} ds. \tag{4.2}$$

Now, we consider the special case where it is additionally assumed that the fractional integration orders are the same for all variables, i.e. $d_j = d$ for $j = 1, \dots, m$. Then (4.2) simplifies to

$$\begin{aligned} G_d(t) &= \frac{1}{\Gamma(d)} \int_0^\infty s^{d-1} e^{A(t-s)} BI_{(0,\infty)}(t-s)\, ds \\ &= \frac{e^{At}}{\Gamma(d)} \int_0^t s^{d-1} U \operatorname{diag}(e^{-\lambda_i s})\, ds\, U^{-1}B = \frac{e^{At}U}{\Gamma(d)} \int_0^t s^{d-1} \operatorname{diag}(e^{-\lambda_i s})\, ds\, U^{-1}B \\ &= \frac{e^{At}U}{\Gamma(d)} \begin{pmatrix} \lambda_1^{-d} \int_0^t s^{d-1} e^{-\lambda_1 s}\, ds & & \\ & \ddots & \\ & & \lambda_m^{-d} \int_0^t s^{d-1} e^{-\lambda_m s}\, ds \end{pmatrix} U^{-1}B \\ &= e^{At}U \begin{pmatrix} \lambda_1^{-d} P(d, \lambda_1 t) & & \\ & \ddots & \\ & & \lambda_m^{-d} P(d, \lambda_m t) \end{pmatrix} U^{-1}B, \end{aligned}$$

where $P(\alpha, x) = \frac{1}{\Gamma(\alpha)} \int_0^x e^{-t} t^{\alpha-1}\, dt$ is the lower incomplete gamma function with complex argument $x \in \mathbb{C}$. It follows from (2.21) and (3.5),

$$\mathbf{O}_t^{d,A,B} = \int_{\mathbb{R}} G_d(t-u)\, \mathbf{L}(du), \quad t \in \mathbb{R}. \tag{4.3}$$

We see that the main advantage of OU processes is that the explicit expression of the fractionally integrated kernel is easy to compute, which is not the case for general MFICARMA processes.

Finally, we would like to mention that the usual definition of an (not fractional) OU process driven by Brownian motion is as the solution of a stochastic differential equation, the so-called Langevin equation. The next proposition shows that this is also true for Lévy-driven multivariate fractional OU processes.

Proposition 4.3. *The process* $\mathbf{O}_t^{d,A,B}$ *as given in* (4.1) *is the unique stationary solution of the SDE of Langevin-type*

$$d\mathbf{O}(t) = A\mathbf{O}(t)\, dt + B\mathbf{M}_d(dt), \quad t > 0, \tag{4.4}$$

where the matrices $A, B \in M_m(\mathbb{R})$ *are defined as in Definition* 4.1.

Proof. Let $t_0 < s < t$. Notice that Eq. (4.4) can be written in the integral form

$$\mathbf{O}(t) - \mathbf{O}(t_0) = \int_{t_0}^t A\mathbf{O}(s)\, ds + B[\mathbf{M}_d(t) - \mathbf{M}_d(t_0)].$$

Therefore, setting $F := \begin{bmatrix} \frac{(u-v)^{d_1-1}}{\Gamma(d_1)} & & 0 \\ & \ddots & \\ 0 & & \frac{(u-v)^{d_m-1}}{\Gamma(d_m)} \end{bmatrix}$ and then using (2.21) and Fubini's theorem we obtain

$$\begin{aligned}
\int_{t_0}^{t} A\mathbf{O}(s)\,ds &= \int_{t_0}^{t} A \int_{-\infty}^{s} e^{A(s-u)} B\,\mathbf{M}_d(du)\,ds \\
&= \int_{t_0}^{t} A \int_{-\infty}^{t_0} e^{A(s-u)} B\,\mathbf{M}_d(du)\,ds + \int_{t_0}^{t} A \int_{t_0}^{s} e^{A(s-u)} B\,\mathbf{M}_d(du)\,ds \\
&= \int_{t_0}^{t} A \int_{-\infty}^{t_0} e^{A(s-t_0)} e^{A(t_0-u)} B\,\mathbf{M}_d(du)\,ds \\
&\quad + \int_{t_0}^{t} A \int_{t_0}^{s} \int_{v}^{\infty} e^{A(s-u)} BF\,du\,\mathbf{L}(dv)\,ds \\
&= \int_{t_0}^{t} Ae^{A(s-t_0)}\mathbf{O}(t_0)\,ds + \int_{t_0}^{t} A \int_{v}^{\infty} \int_{u}^{t} e^{A(s-u)} BF\,ds\,du\,\mathbf{L}(dv) \\
&= [e^{A(t-t_0)} - I_m]\mathbf{O}(t_0) + \int_{t_0}^{t} \int_{v}^{\infty} [e^{A(t-u)} - I_m]BF\,du\,\mathbf{L}(dv) \\
&= [e^{A(t-t_0)} - I_m]\mathbf{O}(t_0) + \int_{t_0}^{t} [e^{A(t-u)} - I_m]B\,\mathbf{M}_d(du) \\
&= \mathbf{O}(t) - \mathbf{O}(t_0) - B[\mathbf{M}_d(t) - \mathbf{M}_d(t_0)].
\end{aligned}$$

The proof of the uniqueness is a simple application of Gronwall's Lemma (see e.g. [15, Theorem 3.1]). □

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